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Probing the strong coupling limit of large N SYM on curved backgrounds

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Abstract: According to the AdS/CFT correspondence, the strong coupling limit of large N $\mathcal{N}=4$ supersymmetric gauge theory at finite temperature is described by asymptotically anti de Sitter black holes. These black holes exist with planar, spherical and hyperbolic horizon geometries. We concentrate on the hyperbolic and spherical cases and probe the associated gauge theories with D3-branes and Wilson loops. The D3-brane probe reproduces the coupling of the scalars in the gauge theory to the background geometry and we find thermal stabilization in the hyperbolic case. We investigate the vacuum expectation value of Wilson loops with particular emphasis on the screening length at finite temperature. We find that the thermal phase transition of the theory on the sphere is not related to screening phenomena.

Keywords: D-branes, Black Holes in String Theory, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

Maldacena's conjecture [1] states that SU(N) $\mathcal{N} = 4$ supersymmetric gauge theory in four dimensions is dual to type IIB string theory on an $AdS_5 \times S^5$ background. This conjecture is obtained by considering a collection of N D3-branes in type IIB string theory. The effective theory describing the physics of the D3-branes is $\mathcal{N} = 4$ SU(N) Yang-Mills. D3-branes can also be described as classical black hole solutions in IIB supergravity. Taking $\alpha' \to 0$ and at the same time keeping the mass of the strings stretched between the D3-branes finite defines the near horizon limit of the supergravity solution. On the other hand one is keeping only the ground states of the open strings. This gives $\mathcal{N} = 4$ gauge theory as the exact theory (not only the effective theory for low energies) and the $AdS_5 \times S^5$ background as the near horizon limit in the supergravity solution.

The same arguments apply not only to extremal but also to near-extremal D3-branes [2]. One obtains then asymptotically anti de Sitter black holes with planar horizon geometry. The euclidean continuation of these black hole solutions is conjectured to describe a thermal state of the $\mathcal{N} = 4$ gauge theory with a temperature corresponding to the Hawking temperature of the black hole [1, 3]. Much work has been devoted to the study of this duality, see e.g. [4] for a review.

The supergravity solution is however only an approximation to string theory and corresponds to the large N and strong coupling limit in the dual gauge theory. A direct comparison of quantities calculated at weak coupling in the gauge theory with the corresponding results in the supergravity theory is therefore rather difficult. However many quantities calculated at strong coupling have been shown to be consistent at the qualitative level with general expectations from gauge theories.

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Among the quantities that have been studied for near-extremal D3-branes are effective potentials obtained through D3-brane probes [5, 6] and vacuum expectation values of Wilson loops corresponding to the action of strings ending on the conformal boundary of the AdS black hole [7, 8, 9, 10, 11].

Black holes in AdS space can have spherical, planar or hyperbolic horizon geometries. Although only the planar black holes can be directly related to D3-branes in a thermal state, one expects that also the spherical and hyperbolic black holes describe gauge theories. More specifically in [3] it was conjectured that spherical black holes describe $\mathcal{N}=4$ gauge theories at finite temperature on S^3. One also expects that AdS black holes with hyperbolic horizon describe $\mathcal{N}=4$ gauge theory on the hyperbolic three-plane H^3. We will further investigate these conjectures. In section 2 we will compute the static potential for a test D3-brane in the background of spherical or hyperbolic AdS black holes. We find that at large distances the static potential reproduces the tree level coupling of the scalars to the background geometry in the gauge theory. Since in the hyperbolic case this coupling induces a negative mass term one would expect the gauge theory to be unstable. The D3-brane potential at finite temperature grows however for small distances up to a maximum value and only for large distances reproduces the negative mass term in the gauge theory. At zero temperature there is however no such stabilization and one finds only the tree level negative mass term.[1]

In section 3 we turn to the properties of Wilson loops. We find screening at finite temperature on the sphere and at zero temperature a potential that is very close to the form of the potential at weak coupling. We also speculate about the relation of the screening length at finite temperature and the thermal phase transition between the spherical AdS black hole and empty AdS space. In the hyperbolic case we find screening even at zero temperature.

2. Probing AdS black holes with D-branes

We begin with the relevant bosonic part of the type IIB action[2]

$$S_{IIB} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(R - \frac{1}{4.5!}F_5^2\right). \tag{2.1}$$

The ten-dimensional spacetime will be the product manifold of S^5 and a five-dimensional AdS black hole. The flux of the five-form field strength through S^5 is quantized in terms of the D3-brane charge as $\int F = 2\kappa_{10}^2\mu_3 N$, with $\kappa_{10}^2 = 64\pi^7 g_s\alpha'^4$ and

[1] Similar behavior of the potential of M2-branes in the background of non-extremal D-branes with hyperbolic horizon has been found in [12].

[2] We take a Yang-Mills term for the five-form action with the convention of imposing self-duality at the level of the equations of motion.

$\mu_3^2 = \pi/\kappa_{10}^2$. The metric of the euclidean AdS black holes is

$$ds^2 = \ell^2 \left[\left(u^2 + k - \frac{\mu}{u^2} \right) d\tau^2 + \frac{du^2}{u^2 + k - \frac{\mu}{u^2}} + u^2 \, d\Sigma_{3,k}^2 \right], \tag{2.2}$$

where ℓ is the radius of curvature of the asymptotic AdS geometry and μ is proportional to the black hole mass. The discrete parameter k can take the values $(1, 0, -1)$ corresponding to the spherical, planar or hyperbolic cases respectively. In the following we will always consider $k = 1$ or $k = -1$. $d\Sigma_{3,k}^2$ denotes then the metric on the unit 3-sphere or 3-hyperboloid

$$\begin{aligned} d\Sigma_{3,1}^2 &= d\theta^2 + \sin^2\theta \, d\Omega_2^2 \, , \\ d\Sigma_{3,-1}^2 &= d\theta^2 + \sinh^2\theta \, d\Omega_2^2 \, . \end{aligned} \tag{2.3}$$

The geometry defined in (2.2) is free of conical singularities if the euclidean time coordinate τ is periodically identified, with period

$$\beta = \frac{2\pi u_+}{2u_+^2 + k}, \tag{2.4}$$

where u_+ is the horizon radius

$$u_+ = \sqrt{-\frac{k}{2} + \sqrt{\frac{1}{4} + \mu}} \, . \tag{2.5}$$

The temperature associated to (2.2) is $T = 1/\beta$. Since the boundary of asymptotically AdS metrics is only defined up to conformal transformations [13], we have chosen (2.2) such that the boundary 3-sphere or 3-hyperboloid has radius one. Although an arbitrary radius R can be set by redefining $u = R\tilde{u}$ and $\tau = \tilde{\tau}/R$, the thermodynamics will only depend on the ratio β_{S^1}/R given by (2.4).

The black hole metrics (2.2) with $k = 1$ only exist for $T \geq T_{min} = \sqrt{2}/\pi$, which corresponds to $\mu = 3/4$. For each temperature bigger than T_{min} there are two solutions for μ, one bigger than $3/4$ and one smaller. We will only consider mass parameters $\mu \geq 3/4$. Black holes with smaller values of μ have negative specific heat. They decay due to Hawking radiation and do not correspond to a stable thermal state in the gauge theory. There is furthermore a phase transition at $\mu_{crit} = 2$, or equivalently

$$T_{crit} = \frac{3}{2\pi} \, . \tag{2.6}$$

For temperatures below T_{crit} the black hole solution is unstable due to tunneling to empty AdS-space [14]. The gauge theory interpretation is that of a confining/deconfining phase transition [13, 3]. The free energy of the black hole solutions scales like N^2, consistent with the expected deconfinement at finite temperatures in the gauge theory. For temperatures $T < T_{crit}$ the field theory dual is empty AdS.

Its free energy scales like N^0, signalling confinement. We will discuss this phase transition in more detail in section 3.

For $k = -1$ the horizon radius is defined even for negative mass parameters down to $\mu = -1/4$. Lorentzian black holes with negative mass parameter have also an inner horizon and a timelike singularity such that their Penrose diagram is that of the Reissner-Nordstrom AdS solution [15]. At the extremal case $\mu = -1/4$ inner and outer horizon coincide. The nature of the horizon changes, it becomes a bifurcate horizon with no particular temperature associated to it. Thus the extremal solution is somewhat similar to empty AdS in the Poincaré patch and can be associated with the zero temperature ground state of $\mathcal{N} = 4$ Yang-Mills on the hyperboloid [16, 17]. The fact that the groundstate is given by a solution with negative mass parameter and its implications concerning the entropy of the corresponding gauge theory have been discussed in [18] and the field theory 1-loop calculation of the free energy has been performed recently in [18, 19].

Taking the metrics (2.2) as input for a Freund-Rubin like ansatz [20] we find that the length scale ℓ is determined by the five-form flux through S^5 as $\ell^4 = 4\pi g_s N \alpha'^2$ and the potential of the five-form field strength is given by

$$A_4 = -\ell^4 u^4 d\tau d\Sigma_{3,k} \,, \tag{2.7}$$

where $d\Sigma_{3,k}$ is the unit volume form on either S^3 or H^3. Notice that the metrics (2.2) are not derived as near-horizon limits of D3-branes on an asymptotically flat space.

$\mathcal{N} = 4$ Yang-Mills theory in flat space has a Coulomb branch of vacua on which the gauge group is broken to a smaller group of equal rank. In D-brane language this translates into forces between parallel extremal flat D3-branes being zero. Separating D3-branes in the transverse space dimensions corresponds to turn on Higgs expectation values in the gauge theory. At finite temperature the field theory develops an effective potential which lifts the Coulomb branch and drives the system back to symmetry restauration [21]. Accordingly, forces between near-extremal D3-branes do not cancel. A symmetry breaking pattern $\mathrm{U}(N+1) \to \mathrm{U}(N) \times \mathrm{U}(1)$ is given in the gravity description by separating a single D3-brane from the rest. The effective action of such a configuration can be computed as the action of the single D3-brane in the background of the other N branes

$$S = T_3 \int \left(\sqrt{\hat{g}} + A_4 \right) , \tag{2.8}$$

where T_3 is the D3-brane tension, $\hat{g}$ the induced metric on the probe world-volume and A_4 is the background value of the type IIB four-form potential. This calculation has been done in [5] and [6] and qualitative agreement with the expectations from gauge theory has been found. In this section we want to extend this analysis to the spherical and hyperbolic cases. The main difference with the flat case is that the $\mathcal{N}=4$ scalars can get mass due to a non-minimal coupling to the boundary geometry,

$$S_{SYM}=\frac{1}{g_{YM}^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}D_\mu\Phi D^\mu\Phi+\frac{1}{2}\xi R\Phi^2+\cdots\right). \tag{2.9}$$

Therefore we expect a non-zero effective potential even at zero temperature. Since the boundary geometry is only defined as a conformal class, the $\mathcal{N}=4$ scalars should be conformally coupled, i.e. $\xi=1/6$.

We consider a test D3-brane at a distance u from the origin. Substituting (2.2) and (2.7) in the Born Infeld action (2.8) we obtain[3]

$$S=\beta T_3V_3\ell^4u^4\left(\sqrt{1+\frac{k}{u^2}-\frac{\mu}{u^4}}-1\right), \tag{2.10}$$

with V_3 being the volume of the unit 3-sphere or 3-hyperboloid. For distances $u\gg 1/T$, the potential $V=TS$ derived from (2.10) becomes

$$V=\frac{k}{4\pi^2}NV_3\,u^2\,, \tag{2.11}$$

where we used $T_3=1/(2\pi)^3g_s=N/2\pi^2\ell^4$ [22]. To compare this to the gauge theory in (2.9) we introduce the 't Hooft coupling $\lambda=2g_{YM}^2N$ and identify it on the the supergravity side through $\ell^4=\lambda\alpha'^2$. The radial coordinate u is then related to the Higgs expectation value by $u/2\pi=\Phi/\sqrt{\lambda}$ [23, 5, 6]. The curvature of a unit 3-sphere or 3-hyperboloid is $R=6k$. Substituting these values in (2.9) with $\xi=1/6$, we obtain precisely (2.11). Thus we recover at large u precisely the tree-level coupling of the scalar fields to the background curvature in the gauge theory. At first glance it seems surprising that this term does not suffer some non-trivial renormalization since supersymmetry is broken by both the background geometry and the temperature. At large u we are however probing the ultraviolet properties of the gauge theory. Since $\mathcal{N}=4$ Yang-Mills on $S^1\times S^3$ or $S^1\times H^3$ has vanishing conformal anomaly, in the large u limit we expect to probe the bare couplings of the $\mathcal{N}=4$ gauge theory. This result is an example for the suggestion that large distances in the bulk correspond to small distance scales on the boundary [24].

It is interesting to analyze the large u behaviour of the potential when the dual gauge theory lives on an space with non-vanishing conformal anomaly. We will consider as an example AdS_5 with S^4 boundary of radius one. The metric is

$$ds^2=\ell^2\left(\frac{du^2}{u^2+1}+u^2\,d\,\Omega_4^2\right). \tag{2.12}$$

[3]In [5, 6] the authors worked with asymptotically flat geometries describing near-extremal 3-branes solutions. In defining the probe action, they subtracted the 3-brane volume at infinity. We are directly working in asymptotically AdS spaces and the D3-brane action at infinity does not become a constant therefore no such subtraction can be done. Our expression for $k=0$ coincides with [5] and up to a constant also with [6].

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Using this metric as input for a Freund-Rubin like ansatz, we obtain its associated four-form potential

$$A_4 = -\ell^4 \left[\left(u^3 - \frac{3}{2} u \right) \sqrt{1+u^2} + \frac{3}{2} \operatorname{arcsh} u \right] d\mathrm{Vol}_{S^4} \,. \tag{2.13}$$

Substituting these values in (2.8) and expanding for large u we get

$$V = \frac{1}{2\pi^2} N V_4 \left(u^2 - \frac{3}{2} \log u \right), \tag{2.14}$$

where V_4 is the volume of S^4. The curvature of a four-sphere of radius one is 12. The first term in (2.14) reproduces the conformal coupling of the scalars to the background curvature with the same identification between u and Φ as before. We associate the second term to the conformal anomaly. The trace anomaly of $\mathcal{N}=4$ on S^4 is $T = 3N^2/8\pi^2$ [25] and thus the effective action picks up a logarithmic term [26]

$$\frac{3}{8\pi^2} N^2 V_4 \log \epsilon \,, \tag{2.15}$$

with ϵ playing the part of the momentum cutoff. The coefficient that multiplies the logarithm in (2.14) is precisely the leading $O(N)$ term in $T_N - T_{N+1}$. The variable u sets the energy scale in the field theory at which the gauge group is Higgsed from $\mathrm{SU}(N+1)$ to $\mathrm{SU}(N)$. It is therefore u what substitutes the cutoff ϵ in (2.14).

The identification of u with the vacuum expectation value of the scalar fields is only correct at large u. It has been argued [23] that in order to compare radial distances in supergravity with energy scales in the gauge theory one should use an isotropic coordinate defined by

$$\rho = \exp \int \frac{du}{u^2 + k - \frac{\mu}{u^2}} = \left(u^2 + \frac{k}{2} + \sqrt{u^4 + k u^2 - \mu} \right)^{1/2}, \tag{2.16}$$

or equivalently

$$u^2 = \frac{\rho^4 + \mu + \frac{1}{4}}{2\rho^2} - \frac{k}{2} \,. \tag{2.17}$$

In terms of the new coordinate ρ, the difference between the D3-brane potential at infinity and at the horizon is

$$V(\rho_+) - V(\infty) = \frac{V_3 N}{4\pi^2} \left(-k \left. \frac{\rho^2}{2} \right|_{\rho\to\infty} + \rho_+^4 - 2k\rho_+^2 \right). \tag{2.18}$$

The action of the euclidean black hole solutions (2.2) has been calculated in [16, 17]. Using their results and our coordinate, we find for the free energy

$$F_N = \frac{V_3 N^2}{8\pi^2} (\rho_+^4 - 2k\rho_+^2) \,. \tag{2.19}$$

If we add one D3-brane the free energy changes by an amount $F_{N+1} - F_N$. The leading $O(N)$ term of this change is precisely given by (2.18) after discarding the quadratically divergent term. The quadratic divergence does not influence the thermodynamics since it is temperature independent. For this reason we do not see this contribution in the change of the free energy.

It is convenient to introduce an "effective temperature" defined by

$$\begin{aligned} \rho_+ &= \pi T_{\mathit{eff}}, \qquad (2.20)\\ T_{\mathit{eff}} &= \frac{T}{\sqrt{2}}\left(1+\sqrt{1-\frac{2k}{\pi^2T^2}}\right)^{1/2}. \qquad (2.21) \end{aligned}$$

In analogy with [5] and [6] we then introduce the scalar mass parameter M by

$$M = \rho - \rho_+ \,. \qquad (2.22)$$

The static D3-brane potential takes now the form

$$V(M) = \frac{\pi^2 T_{\mathit{eff}}^4 V_3 N}{4}\left[1 - \frac{1}{(1+\frac{M}{\pi T_{\mathit{eff}}})^4} + \frac{k}{2\pi^2 T_{\mathit{eff}}^2}\left(\frac{3}{(1+\frac{M}{\pi T_{\mathit{eff}}})^2} + \left(1+\frac{M}{\pi T_{\mathit{eff}}}\right)^2 - 4\right)\right], \qquad (2.23)$$

where we chose to subtract a constant such that it vanishes at the horizon. The first two terms in (2.23) reproduce the potential obtained for planar horizon in [5, 6] by substituting T_{eff} with T.[4] Thus the effect of the curved background on the Higgs potential is to replace T by T_{eff} and add an additional term proportional to the curvature, i.e. the third term in (2.23). The large temperature limit can be alternatively interpreted as the large radius limit for the boundary 3-geometries. Notice that $T_{\mathit{eff}} \to T$ as T tends to infinity. For completeness we give the expansion of the potential for values of the scalar mass $M \ll T_{\mathit{eff}}$

$$V = \frac{\pi^2 T_{\mathit{eff}}^4 V_3 N}{4}\sum_{k=1}^{\infty}\left(\frac{M}{\pi T_{\mathit{eff}}}\right)^k\left(a_k + \frac{k}{2\pi^2 T_{\mathit{eff}}^2} b_k\right), \qquad (2.24)$$

where the coefficients are given by

$$a_k = (-)^k \binom{k+3}{k}, \qquad b_1 = -4\,, \qquad b_2 = 10\,, \qquad b_{k>2} = 3\,(-)^k(k+1)\,. \qquad (2.25)$$

The definition (2.20) of T_{eff} allows us to treat also the case of empty AdS in the global patch by taking $T_{\mathit{eff}} = 1/\sqrt{2}\pi$. The D3-brane potential for $k=1$ at different temperatures as a function of the scalar mass M is shown in fig. 1.

For $k=-1$, the conformal coupling to the curvature of the hyperboloid induces a negative mass term for the $\mathcal{N}=4$ scalars. We have already seen in (2.11) that

[4] Note however that the potentials in [5, 6] have not been chosen to vanish at the horizon.

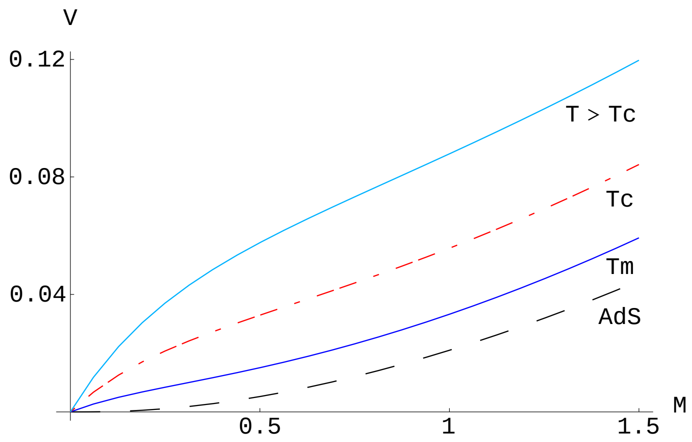

Figure 1: Static D3-brane potential in the background of spherical AdS-black holes for T_{min}, T_{crit} and $T > T_{crit}$. We have also plotted the potential obtained from empty AdS in the global patch.

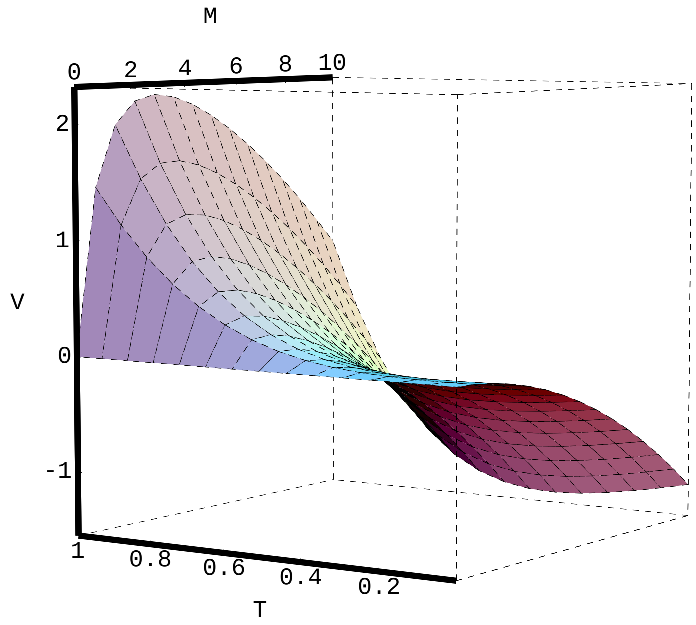

Figure 2: Static D3-brane potential in the background of hyperbolic AdS-black holes (on the vertical axes we have ploted V/NV_3).

this tree level coupling is reproduced in the D3-brane potential. Therefore we would expect the gauge theory to be unstable. An analysis of (2.23) shows however that the static potential grows for small M reaching a maximum at

$$M = \pi T_{\mathit{eff}} \left(\frac{1 + \left(2\pi^2 T_{\mathit{eff}}^2 + \sqrt{4\pi^4 T_{\mathit{eff}}^4 - 1}\right)^{2/3}}{\left(2\pi^2 T_{\mathit{eff}}^2 + \sqrt{4\pi^4 T_{\mathit{eff}}^4 - 1}\right)^{1/3}} \right)^{1/2} - \pi T_{\mathit{eff}} \, . \tag{2.26}$$

This formula is valid also for $T_{\mathit{eff}} < 1/\sqrt{2}\pi$, where $1/\sqrt{2}\pi$ is the effective temperature of empty AdS. The presence of a maximum means that in spite of the negative tree level mass the gauge theory is still driven back to the symmetric phase due to thermal effects as long as the scalar vacuum expectation value is not too large. Of course these statements apply to the limit of large N and large 't Hooft coupling λ, where the supergravity approximation is valid. Notice that the height of the potential maximum is proportional to N. For finite N and coupling the theory could still be unstable e.g. due to tunneling of D3-branes through the potential wall. It would also be interesting to see how much of this behavior is reproduced at weak coupling. For this one would have to do a one-loop calculation of the effective potential of $\mathcal{N}=4$ SYM on the hyperbolic background in the Higgs phase at finite temperature. A precise comparison of the gauge theory result with the results obtained here from supergravity seems problematic due to the well-known gauge dependence of the effective potential. The values of the effective potential at extrema are however gauge independent and can in principle be compared with the D3-brane potential at the maxima (2.26).

For $T_{\mathit{eff}} \to 0$ the maximum of the potential tends to $M=0$. At $T_{\mathit{eff}}=0$ the potential is monotonically decreasing with M. In fact the D3-brane potential at zero temperature is precisely given by the tree level potential. This suggests that the field theory at zero temperature is unstable, a feature that, according to the AdS/CFT conjecture, should be shared by the dual supergravity solution. In [27] a stability analysis of the extremal case has been presented. The authors found however that this geometry is stable against small perturbations. The resolution of this puzzle might go as follows. A D3-brane probe on the extremal hyperbolic black hole will be driven away from the horizon unless it is placed exactly at u_+, the unstable maximum of the potential. Since the length scale of the supergravity solution is $\ell^4 \sim N$, it seems that the extremal geometry is unstable against decreasing ℓ and therefore increasing its curvature. Therefore one should possibly not only consider variations in the metric but also allow for changes of the four-form potential that determines the five-form flux on the S^5.

3. Wilson loops

We will analyze now the behavior of Wilson loops and heavy quark potentials of $\mathcal{N}=4$ Yang-Mills in S^3 and H^3 using the AdS/CFT correspondence. At large N and large 't-Hooft coupling, the expectation value of a Wilson loop C is given in terms of the dual string theory by [7, 8]

$$\langle W(C)\rangle \sim e^{-(S(C)-S_0)}\,, \tag{3.1}$$

where $S(C)$ is the Nambu-Goto action of the fundamental string whose world-sheet ends on the contour C at the $u\to\infty$ boundary and minimizes the action. We will consider a contour describing the world-line of a $q\bar{q}$ pair of very heavy quarks. The action $S(C)$ is infinite due to the contribution of the world-sheet region close to the boundary. This divergence represents the self-energy of the non-dynamical quarks. In AdS language a single external quark is represented by a string extending straight along the radial coordinate into the AdS interior. It is therefore natural to regularize $S(C)$ by subtracting the action of two separated strings stretching down to $u = 0$ in the case of empty AdS [7, 8], or to the horizon for black hole backgrounds [9, 10]. We have denoted this term by S_0 in (3.1).

We will restrict ourselves to the simplified case where q and $\bar{q}$ have equal charge under the $\mathcal{N} = 4$ R-symmetry group, SU(4). This implies that the minimal area configuration will lie at a single point in the S^5 dimensions. We take q and $\bar{q}$ to be static and using the symmetries of the sphere and hyperboloid we place them at $(\theta, \phi, \psi) = (0, 0, 0)$ and $(\theta, 0, 0)$ (for the sphere $\theta \in (0, \pi)$ while for the hyperboloid $\theta \in (0, \infty)$).

The energy of the $q\bar{q}$ configuration is given by the vacuum expectation value of the Wilson loop divided by β. Following [7]–[10] we find[5]

$$E = \frac{\lambda^{1/2}}{\pi z}\left[\int_1^\infty dy\left(\sqrt{\frac{y^4 + kz^2y^2 - \mu z^4}{y^4 + kz^2y^2 - 1 - kz^2}} - 1\right) - 1 + zu_+\right]. \tag{3.2}$$

We have defined $z = 1/u_0$ with u_0 being the minimal radial coordinate of the world-sheet ending on C. The mass of the black hole is related to the effective temperature defined in (2.21) by $\mu = (\pi T_{\mathit{eff}})^4 - 1/4$. As in the previous section, expression (3.2) describes both the cases of an S^3 and H^3 boundary by taking respectively $k = 1$ or $k = -1$. The angular distance between the quarks is given in terms of z by

$$\theta = 2z\sqrt{1 + kz^2 - \mu z^4}\int_1^\infty \frac{dy}{\sqrt{(y^4 + kz^2y^2 - \mu z^4)(y^4 + kz^2y^2 - 1 - kz^2)}}. \tag{3.3}$$

The interaction potential between the pair of static quarks can be read directly from (3.2). If $E \le 0$, $V_{q\bar{q}}(\theta) = E$. When $E > 0$ the configuration with two separated world-sheets extending straight to the AdS interior is energetically favored. Then $V_{q\bar{q}} = 0$ corresponding to screening of the charges [9, 10].

3.1 Quark-antiquark potentials on S^3.

The most interesting aspect of $\mathcal{N} = 4$ Yang-Mills on S^3 is the existence of a phase transition at $T = T_{crit}$ [3]. In the supergravity dual its counterpart is the Hawking-Page phase transition between black hole and pure AdS solutions to the Einstein's equations [14]. The study of quark-antiquark potentials should provide an additional piece of information about the physics involved in it.

[5] Recently deformations of AdS black holes with non-constant dilaton and the quark-antiquark potential in these backgrounds has been studied in [28].

For temperatures below T_{crit} the supergravity dual to $\mathcal{N}=4$ Yang-Mills on S^3 is empty AdS with the euclidean time coordinate τ periodically identified with period $\beta = 1/T$. The low temperature phase corresponds to set $\mu = 0$, $k = 1$ in (3.2) and (3.3), independently of T. Those expressions can be integrated to give[6]

$$\begin{aligned}\theta &= \frac{2z}{\sqrt{(1+z^2)(2+z^2)}}\Big(\Pi(\kappa'^2,\kappa) - K(\kappa)\Big)\,,\\ E &= \frac{\lambda^{1/2}\sqrt{2+z^2}}{\pi z}\Big(\kappa'^2 K(\kappa) - E(k)\Big)\,,\end{aligned} \tag{3.4}$$

where $\kappa = 1/\sqrt{2+z^2}$, $\kappa' = \sqrt{1-\kappa^2}$. The functions $K(\kappa)$, $E(\kappa)$ and $\Pi(\alpha^2,\kappa)$ denote the complete elliptic integrals of first, second and third kind.[7] For small z, which corresponds both to small inter-quark separations and to the large radius limit of S^3, we recover the flat limit result [7, 8]

$$V_{q\bar{q}} = \frac{\lambda^{1/2}}{L\pi}\left(\Pi\left(\frac{1}{2},\frac{1}{\sqrt{2}}\right) - K\left(\frac{1}{\sqrt{2}}\right)\right)\left(K\left(\frac{1}{\sqrt{2}}\right) - 2E\left(\frac{1}{\sqrt{2}}\right)\right) = -\frac{4\pi^2\lambda^{1/2}}{\Gamma(1/4)^4 L}\,. \tag{3.5}$$

Since we are considering 3-geometries of unit radius the inter-quark separation is just $L = \theta$.

The fact that the entropy of empty AdS is zero indicates that the N^2 degrees of freedom of the dual gauge theory are somehow "confined". In [3] it was argued that this is not dynamical confinement but just a kinematical effect: Gauss law on a compact space does not allow for net charges. In agreement with that, fig. 3a shows that the strong coupling quark-antiquark potential has a Coulomb-like behavior. Kinematical confinement is also present at weak coupling since the $\mathcal{N}=4$ gauge theory is in a Coulomb phase at zero temperature.

We will now compare the weak and strong coupling limits of the potential more quantitatively. In the weak coupling and zero temperature limit, the potential between a pair of static charges on S^3 can be calculated explicitly. At small coupling we can neglect the non-linearity of Yang-Mills and just consider an abelian $\mathcal{N}=4$ gauge theory. The potential between a pair of static charges with equal R-charge is due to the interchange of gauge bosons and a linear combination of the six $\mathcal{N}=4$ scalars, i.e. that with R-charge aligned with the quarks R-charge

$$V_w = -\frac{g_{YM}^2 N}{4\pi^2}\left((\pi-\theta)\cot\theta + \frac{(\pi-\theta)}{\sin\theta}\right). \tag{3.6}$$

[6] Quark-antiquark potentials in a Yang-Mills state associated to the uniform distribution of flat 3-branes on a disk have been studied recently in [11]. When the $q\bar{q}$ pair is placed on an axes orthogonal to the disk, the resulting expressions coincide formally with (3.4). Since the Nambu-Goto action depends on products of metric elements, different metrics can produce formally equivalent expressions for the Wilson loops.

[7] We follow the notation in [29].

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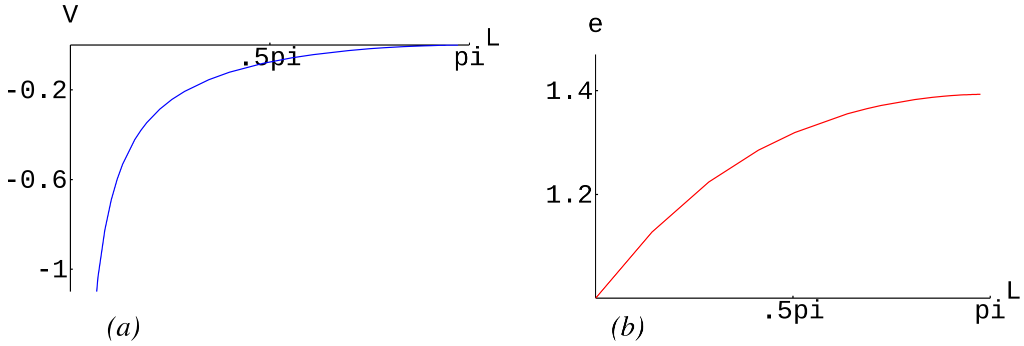

Figure 3: (a) Potential for a pair of static heavy quarks on S^3 on the large N, large 't-Hooft coupling limit (we have set $\lambda = 1$). (b) Normalized ratio between the strong and weak coupling limits of the potential.

The first term is associated to gauge boson interchange. It is obtained by solving the Laplace equation on S^3 with opposite sign delta sources at the positions of the quarks, and deriving from that the energy of the configuration. The second term in (3.6) is due to the scalar interchange. It is obtained by solving the Laplace equation with sources and a mass term since the $\mathcal{N} = 4$ scalars couple to the curvature of the 3-sphere. Indeed we have seen in the previous section that the scalars are conformally coupled, i.e. $m^2 = 1$, as expected [13].

The comparison between the weak and strong coupling limits of the potential at $T = 0$ shows important renormalization effects. As in the flat case [7], the dependence of the potential on the 't-Hooft coupling changes from weak to strong coupling. On S^3, in addition, also its functional dependence on the inter-quark separation changes. The radius of the sphere introduce a new scale and even in the conformally coupled case nothing constrains the dependence of the potential on the angular separation of the quarks. This does not contradict the fact that empty AdS is an exact solution to the string equations of motion [30] since (3.2) and (3.3) can receive α' corrections due to quantum fluctuations of the string ending on the quarks world-line [31]. In order to compare the different θ dependences of (3.4) and (3.6), we have plotted in fig. 3b the quotient between the strong and weak coupling potentials normalized such that it tends to 1 at $\theta \to 0$, i.e. $e = e_0 V_{st}/V_w$ with

$$e_0 = \lim_{\theta \to 0} \frac{V_w(\theta)}{V_{st}(\theta)} = \frac{\Gamma(1/4)^4 \lambda^{1/2}}{2(2\pi)^3}. \tag{3.7}$$

The potential at strong coupling is proportional to $\lambda^{1/2}$ instead of λ, which is the weak coupling result. It is interesting to notice that this reduction in the expected intensity of the potential at strong coupling decreases slightly as the angular separation of the quarks grows.

At $T \geq T_{crit}$ the field theory undergoes a phase transition, detected because its supergravity dual description changes from empty AdS to an AdS Schwarzschild

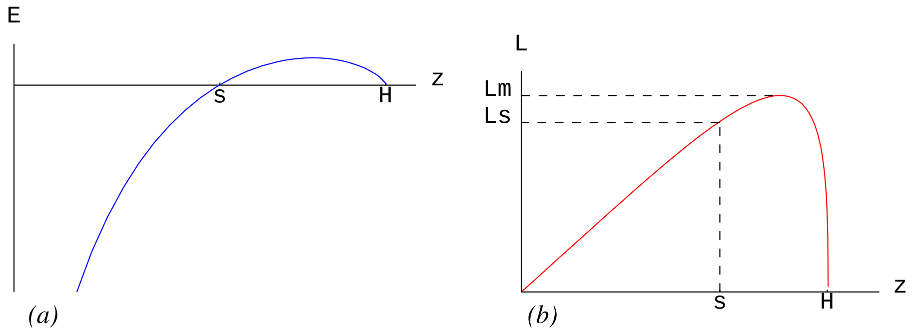

Figure 4: Energy of the single string configuration (a) and quark-antiquark separation (b) as functions of the minimal radial distance that the string reaches in the black hole background. (H denotes the black hole horizon).

black hole background. The black hole solutions have an entropy $S \sim N^2$. This implies that there are $\sim N^2$ degrees of freedom contributing to the entropy of the dual gauge theory and suggests that the gauge theory is in a deconfined phase [3, 13]. Since low temperature confinement is just the statement of Gauss law on the sphere, we could think that what triggers the phase transition is screening. Let us analyze the situation at weak coupling. The $q\bar{q}$ potential at weak coupling is proportional to $\frac{\exp(-m_{el}L)}{L}$. To lowest order we can estimate the electric screening mass to be $m_{el}^2 \propto \lambda T^2$ [21]. If the screening length $1/m_{el}$ is much smaller than the size of the compact manifold the gauge theory will qualitatively behave as in flat space. Charges will be screened and we expect the entropy to scale like N^2. If however the screening length is bigger or of the order of the compact space size, the quark charges are not screened over maximal distances on the compact space. In this case physical states have to be color neutral and no quasi-free charges are possible. Thus we expect the entropy to scale like N^0. Therefore even at weak coupling there will be a phase transition in the large N gauge theory at finite temperature on a compact space. If there is not other dominant mechanism, screening phenomena will induce a phase transition at $T_c \propto 1/\sqrt{\lambda}\pi$ for the case of a three sphere of unit radius where the maximal distance is π. Although at weak coupling this crude estimate gives a very high critical temperature, strong coupling effects could importantly lower it.

We derive again $V_{q\bar{q}}$ from (3.2) and (3.3). The high temperature phase of $\mathcal{N}=4$ on S^3 corresponds to $\mu \geq 2$. The results parallel those obtained in the flat case [9, 10]. The single string configuration with boundary on the quarks positions exists only for inter-quark distances smaller than a certain threshold, $L_m(T)$. For each distance $L < L_m(T)$ there are two string configurations that extremize the Nambu-Goto action. One of them has bigger action than that of two separated strings extending straight to the black hole horizon and is therefore energetically disfavored. For separations $L_s(T) < L \leq L_m(T)$, the action of the second single string configuration is also bigger than that of two separated strings (see fig. 4). Therefore $V_{q\bar{q}} = 0$ for quark separations bigger than $L_s(T)$: the high temperature phase exhibits total screening.

We observe in fig. 5 that the screening length is always much smaller than π. AdS Schwarzschild black hole solutions exist for $T \geq T_{min}$ though they are only stable for temperatures bigger than $T_{crit} > T_{min}$. Black hole solutions with $T_{crit} > T \geq T_{min}$ could correspond to meta-stable states in the dual gauge theory. For completeness, we have also plotted in fig. 5 the potential induced by the black hole solution at T_{min}. The screening length is still very small. Since the supergravity solutions we are treating have constant dilaton, the results obtained for the screening length between electric charges apply also to magnetic and dyonic charges. According to this result the phase transition is not triggered by screening of the electric or magnetic charges. If the degrees of freedom at strong coupling were electrically charged their charges would still be completely screened even at (and below) the critical temperature. Our results seem therefore to indicate that the relevant N^2 degrees of freedom at the phase transition do not carry electric charges. The other possible interpretation is that kinematical confinement of the charges is not the mechanism which triggers the phase transition.

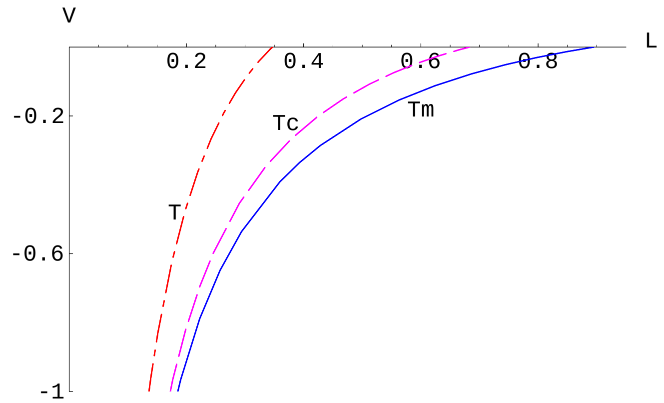

Figure 5: Quark-antiquark potential on S^3 for $T > T_{crit}$, T_{crit} and T_{min} ($\lambda = 1$). In all cases the screening length is much smaller than π.

3.2 Quark-antiquark potentials on H^3.

We turn now to analyze quark-antiquark potentials of $\mathcal{N}$=4 Yang-Mills on the 3-hyperboloid. In the large N and large 't-Hooft coupling limit the inter-quark potential is derived from (3.2) and (3.3) by setting $k = -1$ and $\mu \geq -1/4$; the results are plotted in fig. 6. As in the flat case, for temperatures bigger than zero there is a maximal quark separation after which the charges are completely screened. The surprising result is that even at $T = 0$ the potential has a finite range!

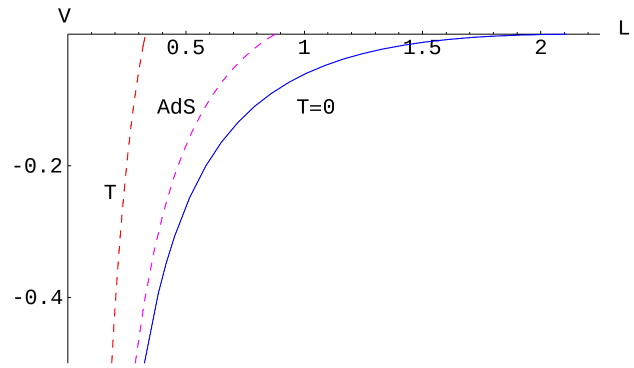

Figure 6: Quark-antiquark potentials on H^3 at $T = 0$, T_{AdS} and $T > T_{AdS}$.

For $T > 0$ the dependence of $V_{q\bar{q}}$ and the inter-quark separation on the minimal radial distance of the single string configuration is as shown in fig. 4. In particular this is true for $T = 1/2\pi$, when the background geometry is just empty AdS with a hyperbolic boundary. The situation at $T = 0$ is however different. The action of the single string configuration is always smaller than that of two separated strings. For each inter-quark separation there is only one single string configuration. As the tip of the string gets closer to the black hole horizon, the inter-quark separation tends to the value

$$L_s = \sqrt{2}\int_0^\infty \frac{dx}{(x+1)\sqrt{x(x+2)}} = \frac{\pi}{\sqrt{2}}\,. \tag{3.8}$$

This relation is obtained by changing variables $y^2 - 1 = (1 - \frac{z^2}{2})\,x$ in (3.3) with $z < 1$ and $\mu = 0$, and then taking the limit $z \to 1$.[8] For separations bigger than L_s the only possible configuration is that of two separated strings, i.e. the charges are screened. Of course since we are on a hyperbolic space, the Coulomb potential will also show exponential falloff at zero temperature in the weak coupling limit, $V \sim \coth\theta$. The result at strong coupling deviates however from this weak coupling behavior since we find a maximum interaction distance. This indicates an additional screening of the charges at strong coupling.[9] The black hole background at $T = 0$ has zero energy but non-zero entropy $S \sim N^2$ [17, 18]. The presence of these N^2 states contributing to the entropy but not to the energy could be related with the presence of total screening.

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[8] As for the low temperature phase on S^3, (3.2) and (3.3) with $k = -1$ and $\mu = -1/4$ can be integrated in terms of elliptic functions. The results coincide formally with those of [11] when the $q\bar{q}$ pair lies in the plane of the disk (see footnote 6).

[9] Screening at $T = 0$ has been also encountered in [11]. The geometry describing that case coincides with the extremal limit of a rotating D3-brane solution [32, 33]. It is interesting to notice that the hyperbolic black hole geometry at $T = 0$ is also extremal.

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